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Cauchy's Functional Equation in the Mean

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IN CELEBRATION OF THE SEVENTIETH BIRTHDAY OF PROFESSOR S. M. ULAM

THEOREM 1. *Let $\alpha \geqslant 1$ be fixed, let $f(x)$ belong to the class $L^{\alpha}(0, z)$ for every $z \geqslant 0$, and satisfy*

$$z^{-2} \int_0^z \int_0^z |f(x+y) - f(x) - f(y)|^{\alpha}\, dx\, dy \to 0, \qquad z \to \infty.$$

Then there is a constant A such that

$$z^{-1} \int_0^z |f(x) - Ax|^{\alpha}\, dx \to 0, \qquad z \to \infty.$$

A stronger hypothesis gives a stronger conclusion:

THEOREM 2. *In the notation of Theorem 1 let*

$$z^{-1} \int_0^z \int_0^z |f(x+y) - f(x) - f(y)|^{\alpha}\, dx\, dy \to 0, \qquad z \to \infty.$$

Then there is a constant A so that $f(x) = Ax$ almost surely for $x \geqslant 0$.

The example $f(x) = e^{-x}$ shows that the hypothesis of Theorem 2 is in a sense best possible.

About forty years ago Professor Ulam asked whether a measurable function $f(x)$ which satisfies the inequality $|f(x+y) - f(x) - f(y)| < \varepsilon$ in the plane must be near to a linear solution of Cauchy's functional equation. This was proved to be the case by Hyers [3], who showed that there is a constant a so that $|f(x) - ax| \leqslant \varepsilon$ holds for all x. Hyers and Ulam [4–7] went on to investigate generalisations and analogues of this situation in a wide range of areas. An account of these matters may be found in the section on *stability* (Chapter VI, pp. 63–69) in Ulam's book of problems [8].

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As Ulam [8] points out (see also Hyers [3]), even if the function $f(x)$ were not measurable it could still be near to a solution of Cauchy's functional equation.

Curiously enough, a similar situation occurred much later in the theory of arithmetic functions. A function $F(n)$, defined on the positive integers, is said to be *additive* if the relation $F(rs)=F(r)+F(s)$ holds whenever the integers r and s have no common factor other than 1, and to be *completely additive* if this relation holds for all positive integers r, s. In this paper [9], Wirsing shows that if an additive arithmetic function $F(n)$ satisfies $|F(n+1)-F(n)| \leqslant C_1$ for some constant C_1 and all $n \geqslant 1$, then there is a completely additive function $G(n)$ and a further constant C_2 so that $|F(n)-G(n)| \leqslant C_2$. He goes on to prove that $G(n)$ must have the form $B \log n$ for some constant B, thus establishing a conjecture of Erdös [2]. The first part of Wirsing's argument and the argument of Hyers [3] are very similar.

The value distribution of arithmetic functions may be considered from a probabilistic point of view. In this theory approximate forms of Cauchy's functional equation again play a rôle. For a discussion of certain of these results see Chapter 11 (Vol. 2) of the author's book [1]. Recent investigations of the author in this area have led to approximate integral equations, also related to Cauchy's functional equation. Since there does not seem to be much literature concerning problems of this last kind, we give two simple examples. The features of their proof(s) are, however, typical.

A proof of Theorem 1 will be given. The same proof method may readily be adapted to establish the following result: *Let*

$$\lambda(z)=\int_0^z \int_0^z |f(x+y)-f(x)-f(y)|^\alpha \, dx \, dy, \qquad z \geqslant 0.$$

Then there is a constant A so that

$$\int_0^z |f(x)-Ax|^\alpha \, dz \leqslant 10^{4\alpha} z^{1+\alpha} \left(\int_z^\infty u^{-2-(2/\alpha)} \lambda(u)^{(1/\alpha)} \, du \right)^\alpha,$$

the final integral being assumed to exist. The value of the constant $10^{4\alpha}$ may be much reduced if desired. If $\lambda(z)=o(z^2)$ as $z \to \infty$ we obtain Theorem 1, and if $\lambda(z)=o(z)$ we obtain Theorem 2.

Proof of Theorem 1. An application of Hölder's inequality shows that

$$\int_0^z \int_0^z |f(x+y)-f(x)-f(y)| \, dx \, dy = o(z^2), \qquad z \to \infty.$$

In particular

$$\iint_{\substack{x+y \leqslant z \\ x, y \geqslant 0}} \{f(x+y)-f(x)-f(y)\} \, dx \, dy = o(z^2), \tag{1}$$

where the region of integration has been reduced from a square to a triangle. This step, which is important in what follows, was first suggested by considerations from the probabilistic theory of numbers.

The integral involving $f(x+y)$ may be treated by the change of variables $x = r(\text{Cos}\,\theta)^2$, $y = r(\text{Sin}\,\theta)^2$, with associated Jacobian of determinant $r\,\text{Sin}\,2\theta$. It becomes

$$\int_{\theta=0}^{\pi/2} \int_{r=0}^{z} f(r)\, r\, \text{Sin}\, 2\theta\, dr\, d\theta = \int_0^z r f(r)\, dr.$$

The choice of a triangular region of integration has permitted a separation of variables.

Typically

$$\iint_{\substack{x+y \leqslant z \\ x, y \geqslant 0}} f(x)\, dx = \int_0^z (z-x) f(x)\, dx$$

and from our Eq. (1) we obtain

$$3 \int_0^z r f(r)\, dr - 2z \int_0^z f(r)\, dr = o(z^2), \qquad z \to \infty. \tag{2}$$

Defining

$$J(w) = \int_0^w f(r)\, dr$$

an integration by parts gives

$$\int_0^z r f(r)\, dr = zJ(z) - \int_0^z J(w)\, dw$$

so that (2) becomes

$$zJ(z) - 3 \int_0^z J(w)\, dw = o(z^2), \qquad z \to \infty. \tag{3}$$

This may be regarded as an approximate differential equation in the variable $\int J(w)$, and an integrating factor of z^{-4} is found so that

$$\frac{d}{dz}\left(z^{-3} \int_0^z J(w)\, dw\right) = o(z^{-2}), \qquad z \to \infty.$$

For $t \geqslant 1$ integrate over the interval $1 \leqslant z \leqslant t$, writing

$$\int_1^t o(z^{-2})\,dz = \int_1^\infty o(z^{-2})\,dz - \int_t^\infty o(z^{-2})\,dz.$$

Then there is a constant D so that

$$t^{-3} \int_0^t J(w)\,dw = D + o(t^{-1})$$

as $t \to \infty$.

Applying this estimate in Eq. (3) we obtain

$$\int_0^z f(r)\,dr = J(z) = 3Dz^2 + o(z), \qquad z \to \infty. \tag{4}$$

From our original hypothesis

$$\int_0^z \left| \int_z^{2z} \{f(x+y) - f(x) - f(y)\}\,dy \right|^\alpha dx$$

$$\leqslant z^{\alpha-1} \int_0^{2z} \int_0^{2z} |f(x+y) - f(x) - f(y)|^\alpha\,dx\,dy = o(z^{1+\alpha}).$$

Here

$$\int_z^{2z} \{f(x+y) - f(x) - f(y)\}\,dy$$

$$= J(2z+x) - J(z+x) - zf(x) - J(2z) + J(z)$$

$$= (6Dx - f(x))\,z + o(z^2),$$

this last step by means of our estimate (4).

With $A = 6D$ Theorem 1 is proved.

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